Math 132: Differential Topology

§ Submersions

<u>Def</u> $f: M \to N$ is called a <u>submersion $p \in M$</u> if df_p is surjective.

If it is a submersion at every point, it is simply called <u>a submersion</u>.

<u>Thm</u> (local submersion theorem)

If $f: M \to N$ is a submersion at $p \in M$, then there exist local coordinates around $p \in M$ and $f(p) \in N$ such that

$$f(x_1, \cdots, x_m) = (x_1, \cdots, x_n).$$

proof) Just as in the proof of local immersion theorem, we start from local parametrizations

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \phi \uparrow & & \uparrow \psi \\ U & \xrightarrow[g]{} & V \end{array}, \qquad U \subset \mathbb{R}^m,\ V \subset \mathbb{R}^n,\ \begin{array}{l}\phi(0) = p, \\ \psi(0) = f(p),\end{array} \qquad g := \psi^{-1} \circ f \circ \phi.$$

By the assumption, $dg_0 : \mathbb{R}^m \to \mathbb{R}^n$ is surjective, so by change of basis in $\mathbb{R}^m$, we may assume $dg_0 = \left.\left(\underbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}_{n} \,\middle|\, \underbrace{0}_{m-n} \right)\right\} n$

Now define $G: U \to \mathbb{R}^m$ by

$$x \longmapsto (g(x), x_{n+1}, \cdots, x_m).$$

Then $dG_0 = \left.\left(\underbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}_{} \right)\right\} m$, and G is a local diffeo at 0.

Note also that $g = \pi \circ G$, where $\pi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$

$$(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_n)$$

is the canonical submersion.

It follows that, after shrinking U sufficiently, we have

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ U \arrow[u, "\phi \circ G^{-1}"] \arrow[r, "\pi"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

as desired. ∎

Cor If f is a submersion at p, then it is a submersion in a neighborhood of p.

That is, submersion is an open condition.

Def For a smooth map $f : M \to N$ of manifolds, a point $q \in N$ is called a regular value for f if $df_p : T_pM \to T_qN$ is surjective at every point $p \in f^{-1}(q)$.

Thm (preimage theorem) If q is a regular value of $f : M \to N$, then the preimage $f^{-1}(q)$ is a submanifold of M with dimension

$$\dim f^{-1}(q) = \dim M - \dim N.$$

proof) For any $p \in f^{-1}(q)$, thanks to the local submersion theorem, we can choose local coordinates around p and q so that

$$f(x_1, \cdots, x_m) = (x_1, \cdots, x_n).$$

Thus, near p, $f^{-1}(q)$ is just the set of points $(0,\cdots,0,x_{n+1},\cdots,x_m)$, and the functions $x_{n+1},\cdots,x_m$ form a local coordinate system on a neighborhood of $p\in f^{-1}(q)$. ∎

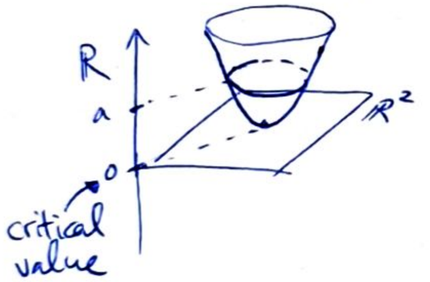
<u>Def</u> A point $q\in N$ that is not a regular value is called a <u>critical value</u>.

Later, we'll learn that the set of critical values is measure zero (Sard's theorem), so almost every point in N is a regular value of f.

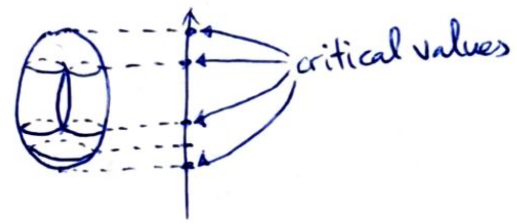
<u>Examples</u>

① $\mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x,y) \longmapsto x^2+y^2$



② $T^2 \longrightarrow \mathbb{R}$



As an application of these notions, let's prove the <u>fundamental theorem of algebra</u>: every non-constant complex polynomial $P(z)$ must have a zero.

First, observe that if M is compact, $q\in N$ is a regular value, and $\dim M = \dim N$, then $f^{-1}(q)$ is a finite set. Moreover, $\# f^{-1}(q)$ is a locally constant function of q (as q ranges only through regular values). (If $p_1,\cdots,p_k$ are points of $f^{-1}(q)$, then we can choose pairwise disjoint neighborhoods $U_1,\cdots,U_k$ mapping diffeomorphically onto $V_1,\cdots,V_k$. We may then take $V = V_1\cap\cdots\cap V_k - f(M-U_1-\cdots-U_k)$.)

4/

proof of the fundamental theorem of algebra)

Consider the unit sphere $S^2 \subset \mathbb{R}^3$ and the stereographic projection

$$h_+ : S^2 - \{(0,0,1)\} \longrightarrow \mathbb{R}^2 \times 0 \subset \mathbb{R}^3$$

say, $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ with $a_0 \neq 0$

We identify $\mathbb{R}^2 \times 0$ with $\mathbb{C}$.

The polynomial map $P : \mathbb{C} \to \mathbb{C}$ corresponds to the map

$$f : S^2 \longrightarrow S^2$$

$$x \longmapsto h_+^{-1} \circ P \circ h_+(x) \quad \text{for } x \neq (0,0,1),$$

$$(0,0,1) \longmapsto (0,0,1).$$

The resulting map is smooth, even in a neighborhood of $(0,0,1)$.

(Let h_- be the stereographic projection from the south pole $(0,0,-1)$.

Then $h_+ h_-^{-1}(z) = \dfrac{z}{|z|^2} = \dfrac{1}{\bar{z}}$, and $h_- \circ f \circ h_-^{-1}(z) = \dfrac{z^n}{\bar{a}_0 + \bar{a}_1 z + \cdots + \bar{a}_n z^n}$ is smooth at 0.)

Moreover, f has only finitely many critical points, since $P'(z)$ is not identically zero.

Hence, the set of regular values of f is connected, and thus $\# f^{-1}(q)$ must be constant on this set. Since $\# f^{-1}(q)$ can't be zero everywhere, it is zero nowhere. Thus f is surjective, and P must have a zero.

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